NONLOCAL MEASURES OF FINITE DEFORMATIONS

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Introduction. Consider a body on whose boundary the displacements, forces, or some other loading conditions are prescribed. As a result, each point of the body will be displaced into a new position. There are two fundamentally different possibilities here: 1) The displacement field will be such that the body undergoes only rigid-body rotation and translation and 2) in addition to displacement of the body as a rigid entity, the shape of the body will be deformed and in the general case the volume will change, i.e., definite deformations will occur in the body.

The problem of describing deformations is one of the basic problems in the mechanics of solids. Different definitions are now used for the very concept of deformations, measures of deformations, and the corresponding mechanical interpretations. They are all of a local character. For example, the definition of the nonlinear Cauchy's strain tensor involves analysis of the change in the distances between pairs of infinitesimally close points. In addition, these pairs lie in a neighborhood of a given point. It is to this point that the corresponding tensor refers. Other definitions are also of a local character and depend on investigation of the kinematics of infinitesimal material volumes of the body [1, 2]. Comparing theoretical constructions to experiments shows that in the theory all "measurements" of deformations occur on an infinitesimal base.

In some cases a more general treatment of this concept is of interest: By deformations we shall mean any quantitative characteristics which describe the difference between the actual field of displacements of points of the body and a set of fields of displacements of the same body as a rigid entity. This approach admits nonlocal definitions, when formations "are measured" on a finite base and their quantitative characteristics refer not to points but rather to the entire body as a whole.

The idea of such an approach appeared in analysis of a specific problem [3]. Let the deformation of the body be conducted in a plane radial matrix. Figures 1a and b display the initial and final, respectively, configurations of the body. How should the deformation of the body as a whole in the process of transformation from the initial to the final configuration be estimated? This can be done as follows. Let the loading parameter be the displacement of the lower boundary h of the body (Fig. 1). We now consider the body to be absolutely rigid and give it a virtual downwards displacements by an amount h. As a result of such a displacement, the configuration of the body will no longer correspond to the form of the matrix. The degree of mismatch can be estimated in terms of the volume of the mismatch regions S_h (Fig. 1c). Regions of mismatch cannot exist in a real process. For this reason, a real process can be interpreted as a process in which the mismatch regions S_h somehow become "smeared" inside the converging channel. Hence it follows that even the deformations of the body as whole can be estimated as the volume of the mismatch regions with respect to the initial volume of the body, i.e., here integral quantitative estimates can be obtained before solving the problem. In [3, 4] this device was employed in order to analyze the flow of free-flowing material under conditions of localization of displacements. It enabled not only giving a quantitative description of the process but also obtaining quantitative estimates.

We shall now attempt to formalize this technique. Let S be the initial configuration of the body and L the boundary of the body. Let the region S transform as a result of deformation into S_t and the boundary L into L_t. We introduce the following notations: $S^0 = S \cap S_t$, $S^- = S \setminus S^0$, $S^+ = S_t \setminus S^0$ (Fig. 2). If we ignore the internal mechanism of the change in shape, then from the external standpoint deformation results in the fact that the parts S^- move away from the region S and parts S^+ are added to the region S. For this reason, the process of deformation of the body as a whole can be judged according to the volume of these regions, their mutual arrangement, and the shape itself. For example, for the case shown in Fig. 2 it is evident that the body is compressed along the direction AB and is stretched along the direction CD.

Here there arises a fundamental circumstance. If rigid-body rotation and translation are imposed on the new configuration of the body, then the internal deformation process will not change as a result. However, the mismatch regions

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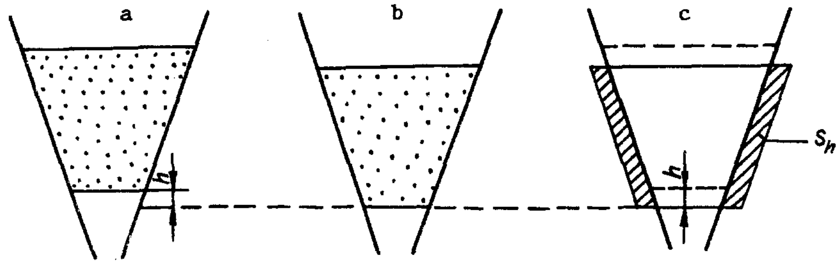

Fig. 1

and their volumes and arrangement also depend quite strongly on this rigid-body displacement. For this reason, the method of [4] cannot be used directly in the general case. We must find something that is common to all situations, something that does not depend on the displacements and rotations of a body as a rigid entity. This can be done by different methods.

First, we consider definitions of deformations which refer to a point of the body but can be "measured" on a nonlocal base. Next we consider a variational definition of deformations. Finally, we consider an extension of the variational approach to the entire body as a whole. For convenience, we confine our attention of the case of a planar deformation.

1. Integral Measures of Deformation. Consider a body S bounded by a contour L. At a definite time the displacements

$$u_1 = u_1(x_1, x_2), \quad u_2 = u_2(x_1, x_2), \tag{1.1}$$

where x_1 and x_2 are Cartesian coordinates of a material point before deformation, occur in this body; u_1 and u_2 are the components of the displacement vector $\mathbf{u}$. The equations (1.1) contain complete information about what it will be natural to term a deformation, but they also contain additional information that has no bearing on deformations. It is obvious that this additional information refers to rotation and displacement of the body as a rigid body. The problem is to separate this information from the equalities (1.1).

We proceed as follows. First, let the displacements (1.1) be realized in the body. Next, we impose on the field (1.1) rigid-body rotation by an angle α and a displacement by the vector $\{C_1, C_2\}$. As a result we obtain the new field:

$$\begin{aligned} \tilde{u}_1(x_1, x_2) &= u_1 \cos\alpha - u_2 \sin\alpha + x_1(\cos\alpha - 1) - x_2 \sin\alpha + C_1, \\ \tilde{u}_2(x_1, x_2) &= u_1 \sin\alpha + u_2 \cos\alpha + x_1 \sin\alpha + x_2(\cos\alpha - 1) + C_2. \end{aligned} \tag{1.2}$$

Thus any prescribed field of displacements $\{u_2, u_2\}$ engenders an infinite set of fields of displacements $\{\tilde{u}_1, \tilde{u}_2\}$. The elements of the set fill the three-dimensional space $\{C_1, C_2, \alpha\}$. From the standpoint of the theory of deformations they all must be indistinguishable. In other words we must find something common to each of the elements (1.2).

Two different approaches are possible here. In any case it will be necessary to give a comparison to classical definitions. For this reason we start with the variant which corresponds to the nonlinear Cauchy–Green strain tensor.

The equations (1.2) describe a vector projection. In a rotation the length of a vector remains unchanged. For this reason, if we set $C_1 = 0$, $C_2 = 0$, in Eq. (1.2), then

$$(\tilde{u}_1 + x_1)^2 + (\tilde{u}_2 + x_2)^2 = (u_1 + x_1)^2 + (u_2 + x_2)^2. \tag{1.3}$$

If the squared length is designated as the norm $\| \mathbf{u} + \mathbf{r} \|$, $\mathbf{r} = \{x_1, x_2\}$, then we can say that all points of the set (1.2), lying on the axis $C_1 = 0$, $C_2 = 0$, have the common invariant $\| \mathbf{u} + \mathbf{r} \|$.

Now the problem is to eliminate from Eq. (1.2) the constants C_1 and C_2 which appear in an additive manner in the equations. For this reason, the equations can be differentiated and in this manner the constants can be eliminated. This operation gives instead of the two equations (1.2) the following four equations ($i, j = 1, 2$):

$$\begin{aligned} \frac{\partial(\tilde{u}_1 + x_1)}{\partial x_i} &= \cos\alpha \frac{\partial(u_1 + x_1)}{\partial x_i} - \sin\alpha \frac{\partial(u_2 + x_2)}{\partial x_i}, \\ \frac{\partial(\tilde{u}_2 + x_2)}{\partial x_j} &= \sin\alpha \frac{\partial(u_1 + x_1)}{\partial x_j} + \cos\alpha \frac{\partial(u_2 + x_2)}{\partial x_j}. \end{aligned} \tag{1.4}$$

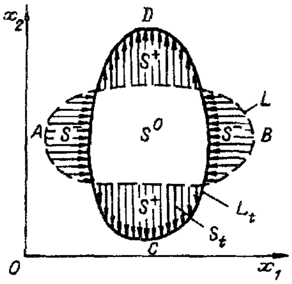

Fig. 2

For Eq. (1.4) the method (1.3) gives two invariants:

$$\left\|\frac{\partial(\mathbf{u}+\mathbf{r})}{\partial x_1}\right\| = \left[\frac{\partial}{\partial x_1}(u_1+x_1)\right]^2 + \left[\frac{\partial}{\partial x_1}(u_2+x_2)\right]^2,$$

$$\left\|\frac{\partial(\mathbf{u}+\mathbf{r})}{\partial x_2}\right\| = \left[\frac{\partial}{\partial x_2}(u_1+x_1)\right]^2 + \left[\frac{\partial}{\partial x_2}(u_2+x_2)\right]^2.$$

In the four equations (1.4) only one parameter a appears on the right-hand side. For this reason, there must exist a third invariant. Since both derivatives $\partial(\mathbf{u} + \mathbf{r})/\partial x_i$ transform according to the rules of vector projection, we can take as the third invariant their scalar product

$$\frac{\partial(\mathbf{u}+\mathbf{r})}{\partial x_1}\cdot\frac{\partial(\mathbf{u}+\mathbf{r})}{\partial x_2} = \frac{\partial(u_1+x_1)}{\partial x_1}\frac{\partial(u_1+x_1)}{\partial x_2} + \frac{\partial(u_2+x_2)}{\partial x_1}\frac{\partial(u_2+x_2)}{\partial x_2}.$$

Any other invariants, for example, the vector product

$$\frac{\partial(\mathbf{u}+\mathbf{r})}{\partial x_1}\times\frac{\partial(\mathbf{u}+\mathbf{r})}{\partial x_2},$$

will be functions of the above-indicated three invariants.

The chosen invariants immediately lead to a nonlinear Cauchy–Green strain tensor ε_{ij}:

$$2\varepsilon_{11} = 2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 = \left\|\frac{\partial(\mathbf{u}+\mathbf{r})}{\partial x_1}\right\| - 1,$$

$$2\varepsilon_{22} = 2\frac{\partial u_2}{\partial x_2} + \left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 = \left\|\frac{\partial(\mathbf{u}+\mathbf{r})}{\partial x_2}\right\| - 1,$$

$$2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} = \frac{\partial(\mathbf{u}+\mathbf{r})}{\partial x_1}\frac{\partial(\mathbf{u}+\mathbf{r})}{\partial x_2}.$$

Thus the Cauchy–Green strain tensor is associated with the procedure of eliminating additive constants with the help of the differentiation operation. It is obvious that this is not the only method of elimination.

We now consider integration, which is in some sense the inverse procedure. We place the origin of coordinates inside the body and integrate the first equation in Eqs. (1.2) over x_1:

$$\int_0^{x_1} (\tilde{u}_1 + x_1)\,dx_1 = \cos\alpha \int_0^{x_1} (u_1 + x_1)\,dx_1 - \sin\alpha \int_0^{x_1} (u_2 + x_2)\,dx_1 + C_1 x_1. \tag{1.5}$$

Now the constant C_1 can be eliminated very simply: for example, Eq. (1.5) is divided by x_1^2, Eq. (1.2) is divided by x_1, and one equation is subtracted from the other. Proceeding in the same manner with the second equation in Eqs. (1.2) and integrals over x_2, we obtain the following four relations:

$$\tilde{w}_{1i} = \cos\alpha w_{1i} - \sin\alpha w_{2i}, \quad \tilde{w}_{2j} = \sin\alpha w_{1j} + \cos\alpha w_{2j}$$

$$\left(w_{ij} = \frac{2(u_i + x_i)}{x_j} - \frac{2}{x_j^2} \int_0^{x_j} (u_i + x_i)\, dx_j \right).$$

Here it is assumed that all integration segments belong to the domain of $\mathbf{u}$. The expressions w_{ij} were chosen so that the conditions $w_{ij} = a_{ij}$ would correspond to the affine transformation

$$u_1 + x_1 = a_{11}x_1 + a_{12}x_2, \quad u_2 + x_2 = a_{21}x_1 + a_{22}x_2,$$

where a_{ij} = const.

It is obvious that the pairs $\{w_{11}, w_{21}\}$ and $\{w_{12}, w_{22}\}$ will be vectors. Hence, the following objective characteristics can be used as measures of deformations:

$$J_{11} = w_{11}^2 + w_{21}^2, \quad J_{22} = w_{12}^2 + w_{22}^2, \quad J_{12} = w_{11}w_{12} + w_{21}w_{22}.$$

We are now ready to make a generalization. Let L_1 and L_2 be linear operators, which transform any constant C into $L_1C = 0$. Then $L_1(\mathbf{u} + \mathbf{r})$, $L_2(\mathbf{u} + \mathbf{r})$ will be vectors and the following characteristics can be taken as operator measures of deformations:

$$\begin{gathered} \| L_1 (\mathbf{u} + \mathbf{r}) \| = [L_1 (u_1 + x_1)]^2 + [L_1 (u_2 + x_2)]^2, \\ \| L_2 (\mathbf{u} + \mathbf{r}) \| = [L_2 (u_1 + x_1)]^2 + [L_2 (u_2 + x_2)]^2, \\ L_1 (\mathbf{u} + \mathbf{r})\, L_2 (\mathbf{u} + \mathbf{r}) = L_1 (u_1 + x_1)\, L_2 (u_1 + x_1) + L_1 (u_2 + x_2)\, L_2 (u_2 + x_2). \end{gathered}$$

As noted above, the operators $L_i = \partial/\partial x_i$ correspond to the nonlinear Cauchy–Green strain tensor. For media with microstructure deformation measures can be constructed on the basis of more complicated operators, for example, $L_i = \partial/\partial x_i + \lambda\partial^2/\partial x_i^2$ (λ is a parameter with the dimension of length), and so on.

2. Variational Definition of Deformations. Thus, only prescribed field of displacements (1.1) engenders a set of fields (1.2). As noted above, all elements of this set must be indistinguishable for any measures of deformations. In the case of the classical definition and the approach studied in Sec. 1, invariant characteristics belonging to all elements of the set (1.2) are calculated. An entirely different approach is possible: Rather than seeking the invariants (1.2), an element which is in some sense distinguished (the base element) is separated from Eq. (1.2). This element itself is now regarded as a characteristic of the deformation process. The requirement of objectivity here indicates only one thing: The base field of displacements must not depend on the starting element from Eq. (1.2). Both the field itself $\{u_1, u_2\}$ and any other field from Eq. (1.2) can be the initial element. The displacements should be the same in each case.

The base field is characterized by the fact that it no longer contains additional rigid-body rotations and displacements. For this reason, for the given field it is now possible to give an objective procedure for making quantitative and qualitative estimates of the type indicated above (see Fig. 2) [5].

Thus in this method the changes in body shape will be characterized by a definite field of displacements. This makes it difficult to compare to existing measures of deformations, because the latter are of a local character. A bridge can be constructed only for the case when the body is small, i.e., it is a small neighborhood of the point (x_1^0, x_2^0).

Thus, let such a body have a field of displacements (1.1). The displacements characterized the propensity of points of the body to change places. If the body is rigid, then the displacements can be made to be zero by choosing appropriate constants C_1, C_2, and α: $\tilde{u}_i \equiv 0$. If the body deforms, then this cannot be done.

It is natural to take as the base situation a situation in which the propensity of points of the body to change places is minimized.

We choose an angle α such that the field $\tilde{u}_1, \tilde{u}_2\,(x_1, x_2)$ would be as close as possible to a constant. Then, by adjusting the constants C_i this field can be made to approach zero. We have no grounds to give preference to any one coordinate x_1 or x_2. For this reason, we choose the simplest and most symmetric norm:

$$\Pi = \left(\frac{\partial \bar{u}_1}{\partial x_1}\right)^2 + \left(\frac{\partial \bar{u}_2}{\partial x_2}\right)^2 + \left(\frac{\partial \bar{u}_1}{\partial x_2}\right)^2 + \left(\frac{\partial \bar{u}_2}{\partial x_1}\right)^2 \underset{\alpha}{\rightarrow} \min. \tag{2.1}$$

The condition $\partial \Pi / 2\alpha = 0$ leads to the result

$$\frac{\partial \bar{u}_1}{\partial x_2} = \frac{\partial \bar{u}_2}{\partial x_1}.$$

Hence

$$\operatorname{tg} \alpha = -\frac{u_{21} - u_{12}}{u_{11} + u_{22} + 2}; \tag{2.2}$$

$$\begin{aligned}
\frac{\partial \bar{u}_1}{\partial x_1} &= \frac{(1 + u_{11})(2 + u_{11} + u_{22}) + u_{21}(u_{21} - u_{12})}{\sqrt{(u_{21} - u_{12})^2 + (2 + u_{11} + u_{22})^2}} - 1, \\
\frac{\partial \bar{u}_1}{\partial x_2} &= \frac{u_{12}(2 + u_{11} + u_{22}) + (1 + u_{22})(u_{21} - u_{12})}{\sqrt{(u_{21} - u_{12})^2 + (2 + u_{11} + u_{22})^2}} = \\
= \frac{\partial \bar{u}_2}{\partial x_1} &= \frac{-(1 + u_{11})(u_{21} - u_{12}) + u_{21}(2 + u_{11} + u_{22})}{\sqrt{(u_{21} - u_{12})^2 + (2 + u_{11} + u_{22})^2}}, \\
\frac{\partial \bar{u}_2}{\partial x_2} &= \frac{-u_{12}(u_{21} - u_{12}) + (1 + u_{22})(2 + u_{11} + u_{22})}{\sqrt{(u_{21} - u_{12})^2 + (2 + u_{11} + u_{22})^2}} - 1,
\end{aligned} \tag{2.3}$$

where $u_{ij} = \partial u_i / \partial x_j$. The base field itself has the form

$$\tilde{u}_i = \frac{\partial \bar{u}_i (x_1^0, x_2^0)}{\partial x_j} x_j.$$

The components (2.3) form a symmetric tensor of rank 2. In the notation of [2, p. 90] they are identical to the components of the tensor $(G_{ij}^{\times 1/2} - \delta_{ij})$, where δ_{ij} is the Kronecker delta function.

Here it is necessary to underscore the following. For finite strain tensors the mechanical interpretation of the tensor components presents a definite difficulty. For the components of small deformations

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

there are no problems, whereas for nonlinear tensors of large deformations everything becomes much more complicated.

We can pose the following question: Does there exist a finite-strain tensor whose components have precisely the same mechanical meaning as the components of an infinitesimal-strain tensor? According to the equations (2.3), the answer is yes: such a tensor exists and it is unique. In this sense the tensor (2.3) has a definite advantage over other finite-strain tensors. In addition, the tensor (2.3) satisfies the variational principle (2.1). The latter circumstance makes it possible to switch to nonlocal generalizations. Here it is convenient to, introduce the concept of a reference body.

3. Method of Reference Fields. Let the reference body be absolutely rigid and have the same shape as the experimental body before a deformation. The field of displacements of the real body has the form (1.1), and the reference field is

$$u_1^0 = x_1 (\cos\beta - 1) - x_2 \sin\beta + B_1, \quad u_2^0 = x_1 \sin\beta + x_2 (\cos\beta - 1) + B_2 \tag{3.1}$$

where β, B_1, and B_2 are constants. For which values of the constants β and B_1 is the displacement field of the reference body closest to the field of real displacements (1.1)? Since we are talking about the closeness of two vector fields, a norm for estimating it must be introduced. It is natural to take a norm which actually transforms into the norm (2.1) as the dimensions of the body decrease to the vicinity of the point (x_1^0, x_2^0). This requirement is satisfied by the following norm (which, by the way, is the simplest norm):

$$\iint_S [(u_1 - u_1^0)^2 + (u_2 - u_2^0)^2]\, dS \rightarrow \min. \tag{3.2}$$

Here S is the region occupied by the body prior to deformation, i.e., the domain of the functions (1.1) and (3.1). We confine our attention to the case when this region is finite, so that all integrals exist and are finite.

All calculations simplify if the origin of coordinates is placed at the center of gravity of the body and the coordinate axes are directed along the principal axes of the section:

$$\iint_S x\,dS = 0, \quad \iint_S y\,dS = 0, \quad J_{xy} = \iint_S xy\,dS = 0.$$

Substituting Eq. (3.1) into Eq. (3.2) and equating to zero the derivatives with respect to B_1, B_2, and β we obtain

$$B_1 = \frac{1}{S}\iint_S u_1(x_1x_2)\,dS, \quad B_2 = \frac{1}{S}\iint_S u_2(x_1x_2)\,dS,$$
$$\operatorname{tg}\beta = \frac{\iint_S (x_1u_2 - x_2u_1)\,dS}{\iint_S (x_1u_1 + x_2u_2 + x_1^2 + x_2^2)\,dS} = -\frac{\iint_S \mathbf{u}\times\mathbf{r}\,dS}{\iint_S (\mathbf{u}+\mathbf{r})\,\mathbf{r}\,dS}, \tag{3.3}$$

where S is the section area. Thus in order to determine the base displacements from the initial equalities (1.1) it is necessary to switch to Eqs. (1.2), setting $C_i = -B_i$ and $\alpha = -\beta$, or the displacement field $\{u_1, u_2\}$ is the base field, if for it all integrals (3.3) vanish.

In a local approach it is not important whether we operate with the values of the displacements in the entire neighborhood of the point (x_1^0, x_2^0) or only on the boundary of such a neighborhood (a locally smooth displacement field is always affine). In a nonlocal approach this is no longer the case. However, all constructions can also be easily made for the case when only the values of the displacements on the boundary are considered. Here, the norm (3.2) can be replaced by the following norm:

$$\int_L [(u_1 - u_1^0)^2 + (u_2 - u_2^0)^2]\,dl \to \min \tag{3.4}$$

where L is the boundary and dl is an element of the boundary. If the coordinate axes are chosen so that

$$\int_L x_1\,dl = 0, \quad \int_L x_2\,dl = 0, \quad \int_L x_1x_2\,dl = 0, \tag{3.5}$$

then

$$B_1 = \frac{1}{L}\int_L u_1\,dl, \quad B_2 = \frac{1}{L}\int_L u_2\,dl,$$
$$\operatorname{tg}\beta = \frac{\int_L (x_1u_2 - x_2u_1)\,dl}{\int_L (x_1u_1 + x_2u_2 + x_1^2 + x_2^2)\,dl} = -\frac{\int_L \mathbf{u}\times\mathbf{r}\,dl}{\int_L (\mathbf{u}+\mathbf{r})\,\mathbf{r}\,dl} \tag{3.6}$$

where $L < \infty$ is the length of the boundary.

We now compare the obtained global formulas to the local formulas. For this we assume that the region S contracts to the point (x_1^0, x_2^0). Then Eqs. (3.3) and (3.6) give

$$B_1 = u_1(x_1^0, x_2^0), \quad B_2 = u_2(x_1^0, x_2^0), \quad \operatorname{tg}\beta = \frac{u_{21} - u_{12}}{2 + u_{11} + u_{22}}. \tag{3.7}$$

The derivation of Eqs. (3.7) employed a restriction on the law of contraction to the point (x_1^0, x_2^0):
in the case (3.3)

$$\int_S x_1^2\,dS \equiv \int_S x_2^2\,dS,$$

in the case (3.6)

$$\int_L x_1^2\,dl \equiv \int_L x_2^2\,dl.$$

It is easy to show that the result (3.7) corresponds to the principle

$$\sum_{i,j=1}^{2}\left[\frac{\partial(u_i-u_i^0)}{\partial x_j}\right]^2 \to \min,$$

i.e., it is actually identical to Eq. (2.1). Thus the nonlinear finite-strain tensor (2.3) corresponds to the nonlocal equations (3.3) or (3.6).

The introduction of the concepts of a reference body makes it possible to take another step in determining nonlocal measures of deformations. The norm (3.4) was employed in order to understand which rigid-body rotation and translation is contained in the initial displacement field. For this reason, the displacement field of the reference body can be regarded as an approximation to the real field in the class (3.1) according to the norm (3.4) or, in other words, the equations (3.3) and (3.6) make it possible to separate from Eq. (1.1) rigid-body rotation and translation. But there is nothing to stop us from taking the next step and thereby separating the general affine deformation.

Let the reference body coincide with the initial body prior to deformation and assume that it undergoes an arbitrary uniform affine deformation

$$u_1^* = b_{11}x_1 + b_{12}x_2 + A_1, \quad u_2^* = b_{21}x_1 + b_{22}x_2 + A_2, \tag{3.8}$$

where u_i^* are components of the displacements of the reference body, and b_{ij} and A_i are arbitrary constants. We choose the constants in Eq. (3.8) so that the field (3.8) is as close as possible to the prescribed field (1.1). The norm

$$\iint_S [(u_1-u_1^*)^2 + (u_2-u_2^*)^2]\, dS \to \min,$$

gives the following result: The field closest to the field (1.1) in the class of affine fields (3.8) is

$$u_1^* = \frac{\iint_S u_1x_1\,dS}{\iint_S x_1^2\,dS}x_1 + \frac{\iint_S u_1x_2\,dS}{\iint_S x_2^2\,dS}x_2 + \frac{\iint_S u_1\,dS}{S},$$
$$u_2^* = \frac{\iint_S u_2x_1\,dS}{\iint_S x_1^2\,dS}x_1 + \frac{\iint_S u_2x_2\,dS}{\iint_S x_2^2\,dS}x_2 + \frac{\iint_S u_2\,dS}{S}. \tag{3.9}$$

Here, as before, the origin of coordinates lies at the center of gravity of the section; Ox_1 and Ox_2 are the principal axes of the section S.

We can proceed similarly, given only information about the displacements on a contour. Let

$$\int_L [(u_1-u_1^*)^2 + (u_2-u_2^*)^2]\, dl \to \min.$$

Then the field closest to the initial field will be

$$u_1^* = \frac{\int_L u_1x_1\,dl}{\int_L x_1^2\,dl}x_1 + \frac{\int_L u_1x_2\,dl}{\int_L x_2^2\,dl}x_2 + \frac{\int_L u_1\,dl}{L},$$
$$u_2^* = \frac{\int_L u_2x_1\,dl}{\int_L x_1^2\,dl}x_1 + \frac{\int_L u_2x_2\,dl}{\int_L x_2^2\,dl}x_2 + \frac{\int_L u_1\,dl}{L}. \tag{3.10}$$

The origin of the coordinates and the axes are chosen so that the equations (3.5) would be satisfied.

We note that now translation and rotation of the body as a rigid entity can be determined in two ways: directly from the formulas (3.3) and (3.6) or first separating the general affine transformation (3.9) and (3.10) and then separating from this transformation rotation and translation according to Eqs. (3.3) and (3.6). It is easy to show that both methods give the same results, i.e., these operations are interchangeable within the constructions made.

Thus from the real and, in the general case, nonlinear field of displacements it is possible to separate a rigid-body rotation, translation, and, moreover, a definite affine transformation. The latter circumstance makes it possible to follow another relation between the local and nonlocal approaches to the concept of deformations. Indeed, locally any transformation can be regarded as affine. For this reason, if it is known that the entire body is subjected to an affine transformation, then the condition of smallness of an elementary volume becomes superfluous for determining the components of the deformations. In this case the tensor can be referred to the entire body as a whole.

The equations (3.9) and (3.10) make it possible to obtain the same result in the general case also. Here, any local formulas for finite deformations can be used in order to determine the tensor. For example, the equations (2.3), where we must make the substitution

$$\frac{\partial u_i}{\partial x_j} \to \frac{\iint\limits_S u_i x_j \, dS}{\iint\limits_S x_j^2 \, dS}.$$

In solving a number of problems the approaches described above are convenient not only for obtaining integral estimates [4, 5] but also for formulating the determining equations [6].

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